Constrained Eigensystem Realization Algorithm for Lightly Damped Distributed Structures

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The temporal correlation method for modal identification of lightly damped distributed structures is extended to include system realization and is shown to be a constrained version of the eigensystem realization algorithm. The method relies on the temporal and spatial orthogonality of the modes of vibration, which are properties of lightly damped or nearly self-adjoint distributed systems. Using these properties, the system realization and modal identification can be performed in the configuration space, as opposed to the state space. In this manner, the computational requirements of the eigensystem realization algorithm are decreased, in addition to constraining the algorithm to take advantage of the lightly damped behavior of the system so that the technique may be less sensitive to sensor noise in the system measurements and provide improved realization capabilities. Analytical and experimental examples illustrate and verify the method. The results are compared with the standard version of the eigensystem realization algorithm.

Introduction

THE eigensystem realization algorithm (ERA) has been gaining popularity for use as a tool for identifying modal parameters of structures. Developed and tested by J.-N. Juang and R. S. Pappa at NASA Langley Research Center, the ERA has been used successfully for minimum-order system realization and modal identification of structures. The ERA was first introduced as an extension of the Ho-Kalman algorithm for constructing a discrete-time state-space representation of a linear system from noisy sensor measurements.[1] In the time domain, the ERA uses the discrete-time state equations $x(k+1) = Ax(k) + Bu(k)$ and the output equation $y = Cx(k)$ and consists of constructing the coefficient matrices A, B, and C using the impulse response. The method is particularly well suited for systems with closely spaced or repeated eigenvalues, as it allows more than a single set of output data generated by different inputs to be used in the technique. This makes it possible to identify eigenfunctions belonging to repeated eigenvalues. The minimum-order system realization consists of determining the system order and then constructing the minimum-order discrete-time state-transition matrix A. The solution of the eigenvalue problem associated with A renders the identified modal parameters. Several recent works have been published by Juang and Pappa[1-6] and Longman and Juang.[7] In addition, Pappa has developed a very useful public domain ERA code for implementation on VAX/VMS systems.

In many ongoing structures/controls experiments, the structure possesses light levels of damping and demands active and/or passive control for vibration suppression. Usually, many sensors are used for system identification and state estimation of the structure to ensure adequate performance of the control system. The degree of noise contamination in the sensors is generally unknown and can make the system realization process especially difficult. Modal survey tests typically use a large number of sensors to adequately define mode shapes, and they can be greater in number than the number of modes to be identified. Using a large number of measurement stations enhances the noise filtering capabilities of the identification as the sensor measurements can use spatial filters (e.g., modal filters) in addition to the temporal filters such as the widely used digital fast Fourier transform. Noise tends to degrade the performance of any identification technique. With this in mind, the technique proposed here constrains the identification to take advantage of the properties of lightly damped systems, in which the response of the system can be described by normal mode behavior. The method is not applicable to general linear systems as the identification renders the normal modal properties. Furthermore, the technique can be used only in cases where the number of sensors is greater than or equal to the number of modes to be identified in the system response.

In this paper, the temporal correlation method (TCM)[8-10] is extended and is shown to be equivalent to a constrained version of the ERA for lightly damped distributed structures. Unlike the ERA, the TCM is based in continuous time and relies on the temporal and spatial orthogonality of the modes of vibration, which are characteristics of lightly damped or nearly self-adjoint systems. The method identifies the natural frequencies and mode shapes from the solution of an nth-order eigenvalue problem, where n represents the number of modes participating in the system response. In addition, in the deterministic setting, the identified eigenvalues of the TCM obey an inclusion principle in which they tend to approach the actual eigenvalues monotonically from above.[10] The nth-order eigenvalue problem has the form $\lambda Mx = Kx$, where M and K are generalized $n \times n$ mass and stiffness matrices, respectively, and the eigensolution is said to be contained in the configuration space.[11] Excluding rigid-body modes, the corresponding modal identification in the ERA requires the solution of the $2n$th-order eigenvalue problem $\lambda x = Ax$. In the free response, a rigid-body mode contributes only one order to the system realization.

The configuration space has received little attention for modal parameter identification of structures. Damping in structures tends to render them non-self-adjoint, which implies that the eigenvalues and eigenvectors are complex quantities as the eigenvalue problem for these systems must be obtained using the state space. In many experiments, however, the damping is light and the overall motion of the structure does exhibit normal mode behavior. In this paper, it is shown that the realization and modal identification can be obtained using the configuration space in cases where the structure is lightly damped. The realization process consists of determining the number of modes in the system response directly, as opposed to the standard version of ERA, which obtains singular values equal to twice the number of participating modes. In

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addition, performing the modal identification in the configuration space has obvious computational advantages over the state space due to the fact that the configuration space has dimension one-half that of the state space.

The parameters identified in lightly damped distributed structures can represent either physical parameters or modal parameters. Structural dynamicists are usually concerned with the later, as the determination of physical parameters relies heavily on the fidelity of the structural model.[12] The first two sections of this paper are devoted to properties of lightly damped distributed structures and spatial discretization for approximating their motion. The next section describes the TCM for modal identification of lightly damped structures. Following this, the TCM is extended and is shown to be a constrained version of the ERA. The identified results are used to form the modal filter[13] from which the modal decay rates can be obtained using the logarithmic decrement.[14] Numerical and experimental results are then presented to illustrate and verify the method.

Equations of Motion for Lightly Damped Distributed Structures

The motion of a lightly damped distributed structure is governed by the partial differential equation (pde) of the form

$$m(P)\ddot{u}(P,t) + \Psi\dot{u}(P,t) + \mathfrak{L}u(P,t) = f(P,t), \quad P \in D \qquad (1)$$

where point P is a point in the domain D of the system, $u(P,t)$ is the displacement at point P and time t, $\mathfrak{L}$ linear self-adjoint differential operator representing the system stiffness, Ψ viscous damping operator, $m(P)$ is a scalar function expressing the mass distribution, and $f(P,t)$ the external force density. The term $\Psi u(P,t)$ is of at least one order of magnitude smaller than the remaining terms in Eq. (1). In addition, the displacement u must satisfy certain geometric and natural boundary conditions. In practice, this pde does not admit closed-form solutions, so that the motion of distributed structures must be approximated by ordinary differential equations.

The differential eigenvalue problem associated with the undamped system is given by

$$\mathfrak{L}\phi_r(P) = \lambda_r m(P)\phi_r(P), \qquad r = 1, 2, \ldots \qquad (2)$$

The solution to Eq. (2) consists of a denumerably infinite set of real non-negative eigenvalues λ_r and associated real eigenfunctions $\phi_r(P)$ that satisfy the boundary conditions of $u(P,t)$ at the boundary of the domain D. The eigenvalues are related to the natural frequencies of undamped vibration by $\lambda_r = \omega_r^2$ $(r = 1,2,\ldots)$. For convenience, the eigenvalues are ordered so that $\lambda_1 \leq \lambda_2 \leq \ldots$. From the expansion theorem,[11] the solution to Eq. (1) can be represented by the infinite summation

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P)q_r(t) \qquad (3)$$

where $q_r(t)$ are generalized coordinates that are modal coordinates for the undamped structure. The inner product between any two functions u and v is defined over the domain R

$$\langle u,v \rangle_s = \int_R uv \, ds, \qquad s \in R \qquad (4)$$

Introducing Eq. (3) into Eq. (1) and following the usual steps

$$\ddot{q}_r(t) + \sum_{s=1}^{\infty} c_{rs}\dot{q}_r(t) + \omega_r^2 q_r(t) = f_r(t), \qquad r = 1, 2, \ldots \qquad (5)$$

where

$$c_{rs} = \langle \phi_s, \Psi\phi_r \rangle_P, \qquad P \in D, \qquad r,s = 1, 2, \ldots \qquad (6)$$

are small viscous damping coefficients and

$$f_r(t) = \langle \phi_r, f(P,t) \rangle_P, \qquad P \in D, \qquad r = 1, 2, \ldots \qquad (7)$$

are modal forces.

Equation (5) represents an infinite set of ordinary differential equations governing the motion of a lightly damped, nearly self-adjoint distributed structure. In the absence of damping, the system is self-adjoint and Eq. (5) is decoupled. High levels of damping tend to destroy the decoupling and the self-adjointness property. In the case of non-self-adjoint systems, it is necessary to put the equations of motion in the standard state-space form and then solve an adjoint eigenvalue problem.[11] Only in special cases does the operator Ψ have $\phi_r(r = 1, 2, \ldots)$ as its eigenfunctions, in which case $c_{rs} = 0$ $(r \neq s)$. When the operator Ψ admits $\phi_r(r = 1, 2, \ldots)$ as its eigenfunctions, the undamped normal mode shapes are identical to the damped mode shapes.[15] In the case of light levels of damping, the self-adjointness property is preserved and the generalized coordinates $q_r(t)(r = 1, 2, \ldots)$ can be regarded as modal coordinates.

Consider the case in which the distributed structure is only lightly damped so that the system can be assumed to be self-adjoint. This implies that the eigenfunctions $\phi_r(r = 1, 2, \ldots)$ are real and orthogonal with respect to the mass and stiffness distributions and can be normalized to satisfy

$$\langle m^{1/2}\phi_r, m^{1/2}\phi_s \rangle_P = \delta_{rs}, \qquad P \in D, \qquad r,s = 1, 2, \ldots \qquad (8)$$

where δ_{rs} is the Kronecker delta function. In addition, the modal coordinates $q_r(t)$ in free vibration are temporally orthogonal and satisfy[10]

$$\lim_{t_f \to \infty} \frac{1}{t_f} \langle q_r(t), q_s(t + \Delta T) \rangle_t = Q_r e^{\zeta_r \omega_r \Delta T} \cos\omega_r \Delta T \delta_{rs}$$

$$t \in [0, t_f), \qquad r,s = 1, 2, \ldots \qquad (9)$$

where Q_r is a positive constant depending on the initial condition $q_r(0)$, ζ_r is the viscous damping factor of the rth mode with $\zeta_r << 1 (r = 1, 2, \ldots)$, ΔT is a time shift, and the damped natural frequencies $\omega_r\sqrt{1 - \zeta_r^2}$ are assumed to be equal to the undamped natural frequencies ω_r.

Spatial Discretization of Distributed Structures

In practice, the identified system is finite order and is used to represent the motion of the distributed structure within certain bandwidths of the frequency response. We consider a discrete or finite-order model of the system. To this end, we define the nth-order approximation to the displacement u given by

$$u^{(n)}(P,t) = \sum_{r=1}^{n} \chi_r(P)u_r(t) = \chi^T(P)u(t) \qquad (10)$$

where the functions $\chi_r(P)(r = 1, 2, \ldots, n)$ must satisfy the geometric boundary conditions. Following the usual steps, the motion of the distributed structure may be approximated by the ordinary differential equation

$$M^{(n)}\ddot{u}(t) + C^{(n)}\dot{u}(t) + K^{(n)}u(t) = F(t) \qquad (11)$$

in which $M^{(n)}$, $C^{(n)}$, and $K^{(n)}$ are $n \times n$ positive definite or semidefinite ($M^{(n)}$ is always positive definite) mass, damping, and stiffness matrices, respectively, and $F(t) = \langle \chi(P), f(P,t) \rangle_P$, $P \in D$. Equation (11) represents the nth-order approximation to the equation of motion of the distributed structure in which the coordinates $u(t)$ considered here are actual physical coordinates of the structure such as rectilinear displacement, rotation, strain, integrated strain, etc. The term $C^{(n)}\dot{u}(t)$ is at least one order of magnitude smaller than the remaining terms in Eq. (11). The undamped algebraic eigenvalue problem associated with Eq. (11) is

$$\lambda_r^{(n)} M^{(n)} u_r^{(n)} = K^{(n)} u_r^{(n)}, \qquad r = 1, 2, \ldots, n \qquad (12)$$

in which the eigenvalues $\lambda_r^{(n)}$ are related to the natural frequencies of vibration by $\lambda_r^{(n)} \cong \omega_r^2 (r = 1, 2, \ldots, n)$, depending on the accuracy of the discretization. The eigenvectors $u_r (r = 1, 2, \ldots, n)$ are orthogonal and can be normalized to satisfy

$$u_r^{(n)^T} M^{(n)} u_s^{(n)} = \delta_{rs}, \qquad s = 1, 2, \ldots \tag{13a}$$

$$u_r^{(n)^T} K^{(n)} u_s^{(n)} = \lambda_r \delta_{rs} \qquad r,s = 1, 2, \ldots \tag{13b}$$

The expansion theorem for discrete systems is then

$$u(t) = \sum_{r=1}^{n} u_r^{(n)} q_r(t) \tag{14}$$

Equation (12) represents the nth-order approximation to the differential eigenvalue problem given by Eq. (2), in which the nth-order approximation to the eigenfunctions is

$$\phi_r^{(n)}(P) = \chi^T(P) u_r, \qquad r = 1, 2, \ldots, n \tag{15}$$

Temporal Correlation Method

Assume that the number of modes participating in the system response is known to be n. It is of interest to identify the modes using n measurements of the system response; i.e., $u(t)$. To this end, we define

$$A^{(n)} = \lim_{t_f \to \infty} \frac{1}{t_f} \langle u(t), u^T(t) \rangle_t, \qquad t \in [0, t_f) \tag{16a}$$

$$B^{(n)} = \lim_{t_f \to \infty} \frac{1}{t_f} \langle u(t), u^T(t + \Delta T) \rangle_t, \qquad t \in [0, t_f) \tag{16b}$$

Next, consider the algebraic eigenvalue problem

$$\gamma_s A^{(n)} w_s = B^{(n)} w_s, \qquad s = 1, 2, \ldots, n \tag{17}$$

The eigenvalue problem given by Eq. (17) is closely related to the eigenvalue problem of Eq. (12). Substituting Eq. (14) into Eqs. (16) and using the temporal orthogonality conditions, Eq. (9), we have

$$A^{(n)} = \sum_{r=1}^{n} Q_r u_r^{(n)} u_r^{(n)^T} \tag{18a}$$

$$B^{(n)} = \sum_{r=1}^{n} Q_r u_r^{(n)} u_r^{(n)^T} e^{\zeta_r \omega_r \Delta T} \cos\omega_r \Delta T \tag{18b}$$

Substituting Eqs. (18) into Eq. (17) yields

$$\left(\sum_r Q_r e^{\zeta_r \omega_r \Delta T} \cos\omega_r \Delta T u_r^{(n)} u_r^{(n)^T} \right) w_s$$

$$= \gamma_s \left(\sum_r Q_r u_r^{(n)} u_r^{(n)^T} \right) w_s, \qquad s = 1, 2,, \ldots, n \tag{19}$$

Using the discrete orthogonality conditions, Eq. (13a), the solutions to Eq. (19) can be shown to be

$$w_r = M^{(n)} u_r^{(n)}, \qquad r = 1, 2, \ldots, n \tag{20a}$$

$$\gamma_r = e^{\zeta_r \omega_r \Delta T} \cos\omega_r \Delta T, \qquad r = 1, 2, \ldots, n \tag{20b}$$

It is evident that the eigensolutions to Eq. (17) provide the nth-order eigenvectors and eigenvalues of the distributed structure and that these quantities are real. The results here apply to the case of light damping so that the natural frequencies can be approximated using

$$\omega_r \cong \frac{1}{\Delta T} \cos^{-1}(\gamma_r), \qquad r = 1, 2, \ldots, n \tag{21}$$

The TCM modal identification proceeds as follows. First, the outer product of n measurements from a distributed structure are temporally correlated, as given by Eqs. (16). In practice, the correlation must be approximated using a finite time t_f. Next, the eigenvalue problem, Eq. (17), is solved to obtain the eigenvectors and eigenvalues given by Eqs. (20). The identified natural frequencies of vibration are then computed using Eq. (21) and the identified nth-order eigenvectors are given by Eq. (20a). In this section, it is assumed that the number of modes participating in the response is equal to the number of output measurements. In the standard ERA, the number of output measurements can be less than the number of identified modes. In the next section, it is shown that the TCM can be extended to become a constrained version of the ERA, in which the number of sensors must be greater than or equal to the number of modes to be identified in the response of a lightly damped structure.

Eigensystem Realization for Lightly Damped Structures

In the preceding section, it was assumed that the number of modes n in the system response was known previous to the identification. Using accuracy indicators to evaluate the effects of noise, the ERA determines the minimum-order realization of the system. The order of this realization is equal to two times the number of modes present in the response excluding rigid-body modes, which contribute only one order per rigid-body mode in the free response to the realization. The minimum-order realization can then be used for modal identification, in addition to system simulation and control design. In the following section, a constrained version of the ERA is shown to perform the realization using a constrained generalized Hankel matrix, and then the modal identification is completed using virtually the same procedure as that used by the ERA.

The ERA has proven to be very useful for system realization, model reduction, and modal identification of linear systems. The first step in the method defines a generalized Hankel matrix composed of rows of Markov parameters, or blocks of system impulse responses at discrete times. The generalized Hankel matrix is then decomposed using the singular value decomposition (SVD) from which the system realization provides a minimum-order realization by retaining only the singular values and vectors that have deterministic significance. The minimum-order realization is computed using a block-shifted generalized Hankel matrix. Then, $2n$ complex modal quantities are extracted from the eigensolution of a $2n$-order realized state-transition matrix. In the constrained version of the ERA to be described, the identified modal quantities are real as the system exhibits near normal mode behavior and they can be obtained in the configuration space.

Consider the output equation

$$y(t) = \langle z, u(P,t) \rangle_P, \qquad P \in D \tag{22}$$

in which $y(t) = [y_1(t)\ y_2(t) \ldots y_p(t)]^T$ is a p vector of output measurements of the distributed system, and $z = [z_1\ z_2 \ldots z_p]^T$ is a p vector of linear operators. In the case where accelerometers at locations $P_i (i = 1, 2, \ldots, p)$ are used as sensor measurements, then $z_i = \delta(P - P_i)\,(d^2/dt^2)$, $(i = 1, 2, \ldots, p)$, so that the output equation at time $t = t_k$ becomes

$$y(t_k) = [\ddot{u}(P_1,t_k)\ \ddot{u}(P_2,t_k) \ldots \ddot{u}(P_p,t_k)]^T \tag{23}$$

The space spanned by u and $y(t)$ is n dimensional because $n < p$ normal modes are contributing to the system response. Next, consider only a single row of Markov parameters in the $p \times ms$ Hankel matrix given by

$$H_c(k) = [Y(t_k)\ Y(t_{k+1}) \ldots Y(t_{k+s-1})] \tag{24}$$

where $t_{k+1} = t_k + \Delta t$, s is the total number of sampling times separated by the sampling time Δt, and $Y(t_k)$ is a Markov parameter comprised of m p-dimensional vectors of impulse responses given by

$$Y(t_k) = [y_1(t_k)\ y_2(t_k) \cdots y_m(t_k)] \tag{25}$$

For the case at hand, it is assumed that the number of output measurements p and the number of sampling times s multiplied by the number of impulse responses m are both greater than the number of modes n contributing to the system response. In this case, the rank of H_c is at least n. In the standard version of the ERA, the generalized Hankel matrix must have rank $2n$. Note that the Hankel matrix may include more than a single initial condition ($m = 1$) at a time so that it will not miss repeated eigenvalues.[1]

The eigenvalue problem given by Eq. (17) together with Eqs. (16) can be reformulated as a constrained version of the ERA for lightly damped systems. The temporal inner products in Eqs. (16) are truncated using

$$A^{(n)} \cong H_c(k)H_c^T(k)\Delta t = \sum_{i=1}^{m}\sum_{j=k}^{s} y_i(t_j)y_i^T(t_j)\Delta t \tag{26a}$$

$$B^{(n)} \cong H_c(k+1)H_c^T(k)\Delta t = \sum_{i=1}^{m}\sum_{j=k}^{s} y_i(t_j)y_i^T(t_{j+1})\Delta t \tag{26b}$$

in which the $p \times p$ matrices $A^{(n)}$ and $B^{(n)}$ include explicitly m impulse responses. Note that the matrices $A^{(n)}$ and $B^{(n)}$ have rank n and are singular for the case in which $p > n$. Hence, with no loss of generality, we have redefined the matrices $A^{(n)}$ and $B^{(n)}$. The truncation of the time histories used in Eqs. (26) as an approximation to Eq. (16) provides meaningful results so long as the time t_f used is at least four times the period of the fundamental period; i.e., $(s - k) > 8\pi/(\omega_1\Delta t)$. In addition, the sampling time Δt must be smaller than one-sixth the period of the highest frequency to be identified; i.e., $\Delta t < 2\pi/(6\omega_n)$.[10]

As in the ERA, the SVD is used as a numerical tool in the system realization. Consider $H_c = USV^T$ in which U and V are $p \times p$ and $ms \times ms$ orthogonal matrices, respectively, and S is a diagonal rectangular matrix of singular values. The rank of H_c can be determined by testing the singular values relative to the desired accuracy.[1] We define

$$H_c = PDQ^T \tag{27}$$

in which D is a diagonal square matrix of selected singular values and P and Q are $p \times n$ and $ms \times n$ isometric matrices (all of the columns are orthogonal: $P^TP = 1$ and $Q^TQ = 1$), respectively. In the constrained version of the ERA, the number of selected singular values n is equal to the number of participating modes and not the system order $2n$. Substituting Eq. (27) into Eqs. (26) and the result into Eq. (17) yields

$$\gamma_s PDQ^TQDP^Tw_s = H_c(k+1)QDP^Tw_s, \qquad s = 1, 2, \ldots, n \tag{28}$$

Invoking $Q^TQ = I$, premultiplying the result by $D^{-1/2}P^T$, and using the definition

$$v_s = D^{3/2}P^Tw_s \tag{29}$$

Eq. (28) becomes

$$\gamma_s v_s = D^{-1/2}P^TH_c(k+1)QD^{-1/2}v_s, \quad s = 1, 2, \ldots, n \tag{30}$$

which is the identical form of the eigenvalue problem developed in Ref. 1, except that, in the development here, $D^{-1/2}P^TH_c(k+1)QD^{-1/2}$ is an $n \times n$ rather than $2n \times 2n$ matrix. In addition, the identified eigenvalues are real and the identified eigenvectors are real and orthogonal, depending on the approximation used in Eqs. (26). The solutions to Eq. (30) provide the natural frequencies given by $\omega_s \cong \cos^{-1}(\gamma_s)/\Delta t$ ($s = 1, 2, \ldots, n$) and, from Eqs. (20a) and (29), the unnormalized mode shapes w_s are given by

$$w_s = M^{(p)}u_r^{(p)} = PD^{-3/2}v_s, \qquad s = 1, 2, \ldots, n; \quad p \geq n \tag{31}$$

in which γ_s and $w_s (s = 1, 2, \ldots, n)$ are the pth-order approximations to the eigenvalues and eigenfunctions, respectively, of the distributed system. The constrained version of ERA requires the use of the SVD to perform the identification. If the system order is known in advance, then the modal identification can be performed directly from the solution of an eigenvalue problem given by Eq. (17) without using the SVD.[8]

Identification of the Modal Decay Rates

In the preceding section, a constrained version of the ERA was shown to provide pth-order approximations to n natural frequencies and unnormalized mode shapes for lightly damped distributed structures. In some instances, it may also be required to identify the corresponding modal decay rates. To this end, we consider the modal filter[13] given by

$$q_r(t) = \langle m(P)\phi_r(P),\, u(P,t)\rangle_P, \qquad P \in D \tag{32}$$

Substituting the pth-order approximation to the displacements and mode shapes given by Eqs. (10) and (15), respectively, into Eq. (32) and using that $M^{(n)} = \langle m(P)\chi(P),\, \chi^T(P)\rangle_P$, $P \in D$, yields the pth-order approximation to the modal filter

$$q_r^{(p)}(t) = u_r^{(p)T}M^{(p)}u(t), \qquad r = 1, 2, \ldots, n \tag{33}$$

and using Eq. (31), we have

$$q_r^{(p)}(t) = w_r^Tu(t) = v_r^TD^{-3/2}P^Tu(t), \qquad r = 1, 2, \ldots, n \tag{34}$$

The logarithmic decrement δ_r for the rth mode is given by

$$\delta_r = \ln\left[\frac{q_r^{(p)}(t_1)}{q_r^{(p)}(t_2)}\right], \qquad t_2 = t_1 + \Delta\tau, \quad r = 1, 2, \ldots, n \tag{35}$$

Then, small viscous damping factors can be computed from[14]

$$\zeta_r \cong \frac{\delta_r}{2\pi k}, \qquad r = 1, 2, \ldots, n \tag{36}$$

in which k is an integer indicating the number of cycles completed in the time period $\Delta\tau$ in Eq. (35).

Numerical Example

To illustrate and compare the ERA and the constrained ERA techniques, the modal parameters associated with the bending vibration of a uniform simply supported beam are identified. The stiffness operator for a beam in bending vibration is given by[11]

$$\mathcal{L} = \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2}{\partial x^2}\right], \qquad 0 < x < L \tag{37}$$

where EI denotes the flexural rigidity of the beam. The geometric boundary conditions for the simple beam require that the ends of the beam undergo zero displacement. For the uniform simple beam, we choose $m = 1$ kg/m, $EI = 50$ Nm^2, and $L = 10$ m. The eigenvalue problem, Eq. (2), admits the closed-form solution[11]

$$\lambda_r = \omega_r^2 = 50\left(\frac{r\pi}{L}\right)^4 \tag{38a}$$

$$\phi_r(x) = \sqrt{\frac{1}{5}}\sin\left(\frac{r\pi x}{L}\right), \qquad 0 < x < L, \quad r = 1, 2, \ldots \tag{38b}$$

We wish to identify the three lowest eigenvalues and corresponding eigenfunctions using six discrete sensors measuring transverse acceleration located at $x_i = iL/7$ ($i = 1, 2, \ldots, 6$). The simulated response is computed using the response of the first three modes given initial modal amplitudes $q_r(0) = 0.001m$ ($r = 1, 2, 3$). We compare the modal identification results of both the ERA and the constrained ERA. In this example, no eigenvalues are repeated so that only a single set of free response data is required to perform the identification.

To this end, 500 samples of data were used with sampling period $\Delta t = 0.10$ s, so that $t_f = 50$ s of data was used. With $p = 6$ in Eq. (23), $s = 500$ in Eq. (24), and $m = 1$ in Eq. (25), the constrained Hankel matrix given by Eq. (24) has dimension 6×500. Note that the free response data fits the requirements for the sampling period and total number of samples; i.e., $\Delta t = 0.10$ s $< 2\pi/(6\omega_3)$ and $s = 500 > 8\pi/(\omega_1 \Delta t)$, respectively. The standard version of the ERA requires more than a single row of Markov parameters so that we choose the generalized Hankel matrix H to be given by

$$H = \begin{bmatrix} H_c(0) \\ H_c(1) \end{bmatrix} \tag{39}$$

To illustrate the effects of the truncation of the temporal inner product, the sensor noise and the system damping on the constrained ERA, two cases are considered. In the first case, the simple beam has no damping and the six accelerometers contain zero noise. In the second case, we add proportional viscous damping to the beam such that $\zeta_r = 0.01$ $(r = 1,2,3)$ and add Gaussian noise to the accelerometers such that, at one standard deviation, the noise added is 2% of the measurement signal. In both cases, the constrained ERA performs the system realization by computing the SVD of $H_c(0)$ given by Eq. (24) [with $p = 6$ and $m = 1$ in Eqs. (23) and (25), respectively], and then by retaining $n = 3$ singular values in the matrix D in Eq. (27). Then, the $n = 3$ order eigenvalue problem given by Eq. (30) is solved with $k = 0$. The natural frequencies and sixth-order approximations to the eigenfunctions are then computed using Eq. (21) ($\Delta T = 0.10$ s) and Eq. (31) ($p = 6$), respectively. The system realization and modal identification for the general case of the ERA is completed in the same manner, except that H in Eq. (39) replaces H_c in Eqs. (27) and (30), in which the eigenvalue problem in Eq. (30) is $2n = 6$ order. Furthermore, the eigenvalues $\gamma_r (r = 1, 2, \ldots, 6)$ in Eq. (30) represent the discrete-time poles of the system from which the continuous-time poles are obtained using $\lambda_r = \ell n(\gamma_r)/\Delta t (r = 1, 2, \ldots, 6)$.[1] The results of the two cases are presented in Tables 1 and 2, respectively. As shown in Table 1, the ERA identifies the modal parameters exactly, whereas in the constrained ERA, we note some discrepancy due to the truncation of the temporal inner product, i.e., Eqs. (26). As expected, noise in the sensor measurements degrades the identification results, as shown in Table 2, with the lowest identified frequency being most sensitive to the noise. Indeed, the noise more readily degrades the contribution of the smaller signals in the accelerometer, which are the lowest frequencies of vibration, because, in this example, the modes have roughly the same amplitude of vibration implying that the higher modes have much larger acceleration. Furthermore, the constrained ERA was able to roughly estimate the first mode in the presence of large sensor noise, which indicates that constraining the identification may provide improved results. The identified sixth-order approximations to the eigenfunctions are presented in Figs. 1–3 for case 2. The identified eigenvectors are normalized so that the largest value was equal to 1. The identified eigenvectors for both the ERA and the constrained ERA agree quite well. It is also interesting to note that the ERA identifies the first mode shape fairly well, although the corresponding eigenvalue is poorly estimated.

To identify the viscous damping in the simple beam using the constrained ERA, we implement the $p = 6$ order approximation to the modal filter given by Eq. (34). The modal time history for the third mode of the simple beam for case 2 is shown in Fig. 4. The viscous damping factor for the third mode can be estimated using Eqs. (35) and (36). To this end, we choose $t_1 = 5$, $\Delta t = 10$ s, and $k = 5$, from which Eqs. (35) and (36) provide $\zeta_3 \cong 0.011$. Furthermore, following the same procedure for modes 1 and 2 provides $\zeta_1 \cong 0.020$ and

Table 1 Modal identification of a simple beam (noise = 0%, $\zeta_r = 0$, $r = 1, 2, 3$)

r	Actual frequencies ω_r, rad/s	Constrained ERA ω_r, rad/s	ERA ω_r, rad/s
1	0.6979	0.7000	0.6979
2	2.7915	2.7914	2.7915
3	6.2810	6.2813	6.2810

Table 2 Modal identification of a simple beam (noise = 2%, $\zeta_r = 0.01$, $r = 1, 2, 3$)

	Actual parameters		Constrained ERA		ERA	
r	ζ_r	ω_r, rad/s	ζ_r	ω_r, rad/s	ζ_r	ω_r, rad/s
1	0.0100	0.6979	0.020	1.771	—[a]	
2	0.0100	2.7915	0.012	2.855	0.047	2.810
3	0.0100	6.2810	0.011	6.285	0.011	6.282

[a]Identified real positive poles.

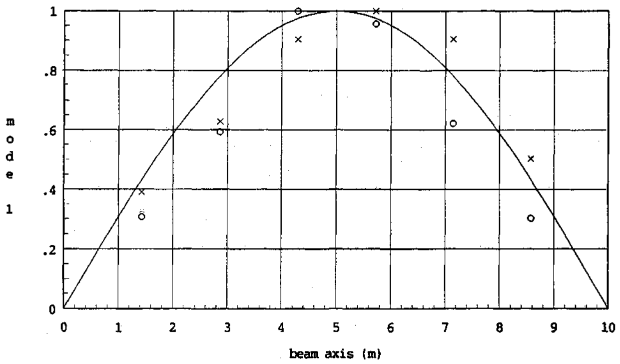

Fig. 1 Identified and actual mode shape 1 for the simple beam: x—ERA; o—constrained ERA.

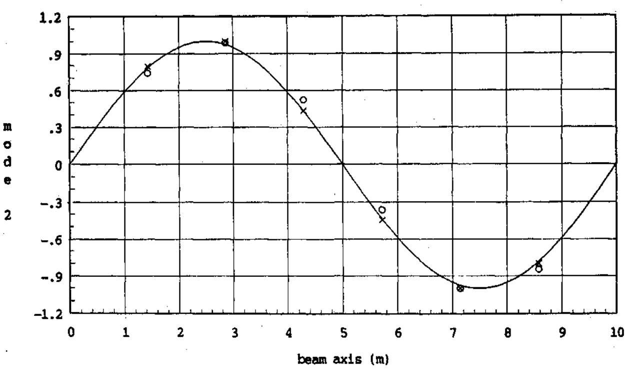

Fig. 2 Identified and actual mode shape 2 for the simple beam: x—ERA; o—constrained ERA.

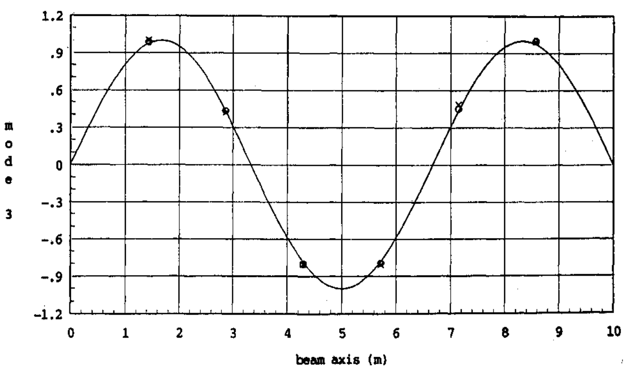

Fig. 3 Identified and actual mode shape 3 for the simple beam: x—ERA; o—constrained ERA.

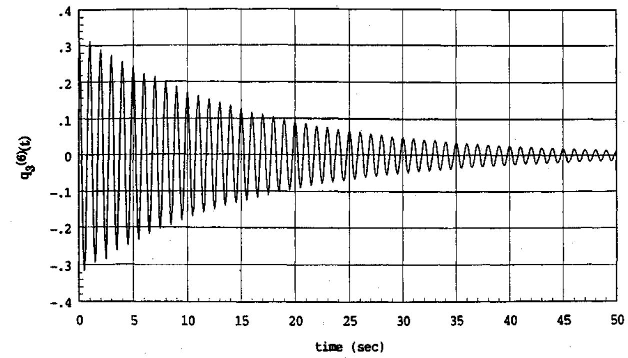

Fig. 4 Mode 3 time history for case 2 of the simple beam using the identified modal filter for mode 3 (noise = 2%).

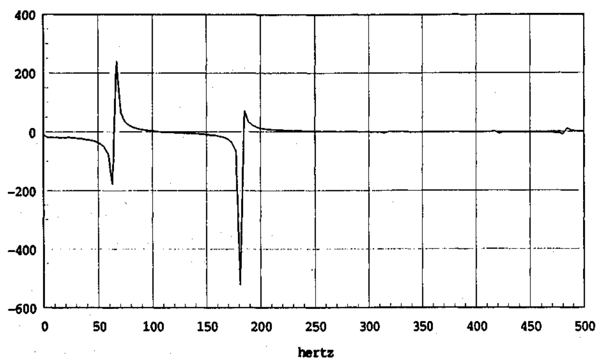

Fig. 5 Real part of a typical FRF of the free-free beam (experimental data).

$\zeta_2 \cong 0.012$. Note that these results are more accurate predictions of the damping than those predicted by the ERA, as shown in Table 2.

Experimental Results

Experimental testing of the constrained algorithm was conducted using data obtained from a 36-in. aluminum beam with cross section 1 × ½ in. Four concentrated masses were placed along the beam 12 in. apart. The beam was suspended vertically by an elastic cord, and four accelerometers were used to measure the system response at the locations of the concentrated masses in a direction normal to the vertical. Impact tests were conducted to obtain four frequency response functions (FRFs). The impulse response functions for the ERA and the constrained ERA techniques were obtained from the inverse Fourier transform of the FRFs. A typical FRF is shown in Fig. 5. Note that two modes dominate in the system response below 500 Hz. In addition, note that the system contains low damping.

Results from the ERA and the constrained ERA were compared using the same procedure used in the numerical example of the simple beam. We used 400 samples of data with sampling period $\Delta t = 0.001$ s so that $t_f = 0.4$, and with four sensor measurements, the constrained Hankel matrix $H_c(0)$ given by Eq. (24) [with $p = 4$ and $m = 1$ in Eqs. (23) and (25), respectively] has dimension 4 × 400. The constrained ERA performs the system realization by retaining $n = 2$ singular values from the SVD of $H_c(0)$. The singular values generated by the two algorithms are displayed in Table 3. The constrained ERA provides four singular values, whereas the ERA provides eight due to the sizes of their respective Hankel matrices. Note that it is easier to determine the number of modes in the system response using the constrained ERA; i.e., compare the ratio 32.1:1.03 to 9.39:1.15. The identified eigenvalues of the constrained ERA are obtained by solving the eigenvalue problem given by Eq. (30) with $k = 0$. For the case of the ERA, using the SVD of the 8 × 400 generalized Hankel matrix in Eq. (39), a $2n = 4$ order eigenvalue problem is solved to obtain the identified discrete-time poles. The identified damping and natural frequencies for both algorithms are presented in Table 4. Comparing the FRF in Fig. 5 with the identified results shown in Table 4, it is evident that both techniques provide excellent estimates of the natural frequencies. To identify the damping in the test article using the constrained ERA, a $p = 4$ order approximation to the modal filter given by Eq. (34) was implemented. The modal time histories for the two modes are shown in Figs. 6 and 7 and they reveal that a smaller sampling time would have improved the time histories, but estimates for the damping can still be obtained. With $t_1 = 0.03$ s, $\Delta t = 0.24$ s,

Table 3 Singular values of the constrained and generalized Hankel matrices (experimental results)

Constrained ERA	ERA
40.9	49.3
32.1	43.9
1.03	30.9
0.744	9.39
	1.15
	1.06
	0.687
	0.509

Table 4 Modal identification of a free-free beam (experimental results)

	Constrained ERA		ERA	
r	ζ_r	$\omega_r/2\pi$, Hz	ζ_r	$\omega_r/2\pi$, Hz
1	0.0040	66.8	0.0561	66.0
2	0.0017	181.0	0.0018	181.0

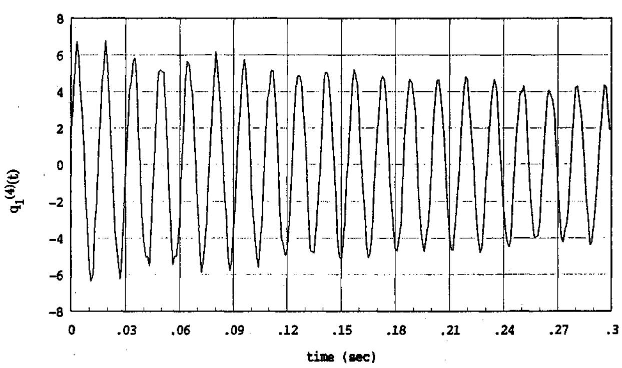

Fig. 6 Mode 1 time history of the free-free beam using the identified modal filter for mode 1 (experimental result).

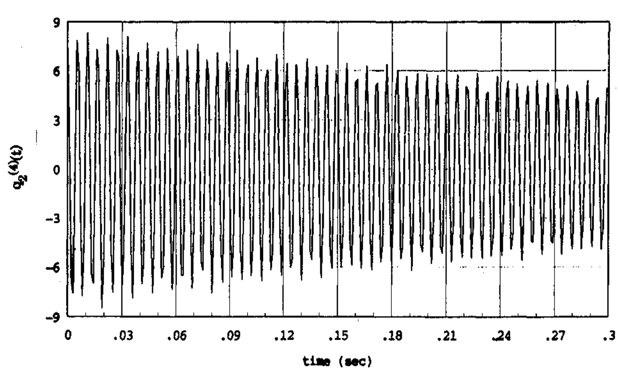

Fig. 7 Mode 2 time history of the free-free beam using the identified modal filter for mode 2 (experimental result).

and $k = 16$ and 43 for Figs. 6 and 7, respectively, the damping was estimated using Eqs. (35) and (36) and the estimates are shown in Table 4. There is a large discrepancy between the two methods in the estimate of the first modal decay rate. In general, the ERA tends to have difficulty identifying modal decay rates for the lowest natural frequencies. The estimated mode shapes (not shown) for both techniques agree quite well.

Conclusions

The temporal correlation method is extended and is shown to be a constrained version of the eigensystem realization algorithm. The method can be used in situations in which the system is lightly damped and the number of sensor measurements is greater than or equal to the number of modes participating in the system response. The constrained version of the eigensystem realization algorithm invokes the behavior of lightly damped distributed structures, in which the response is governed by normal mode behavior. Both analytical and experimental results illustrate and verify the technique and the results are compared with those obtained using the standard eigensystem realization algorithm. The estimates of the natural frequencies and mode shapes for both techniques are quite good. Damping tends to be more difficult to identify accurately; however, the use of the modal filter in the constrained version in conjunction with the logarithmic decrement provides excellent results. Furthermore, the constrained version of the eigensystem realization algorithm requires the solution of an nth-order eigenvalue problem as compared to a $2n$th order in the standard version. In addition, as shown in the experimental results, the larger separation in the singular values of the constrained Hankel matrix as compared to the generalized Hankel matrix may indicate improved realization capabilities in the constrained version of the eigensystem realization algorithm.

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